

## On an elliptic Kirchhoff-type problem depending on two parameters

Biagio Ricceri

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**Abstract** In this paper, on a bounded domain  $\Omega \subset \mathbf{R}^n$ , we consider a non-local problem of the type

$$\begin{cases} -K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under rather general assumptions on  $K$  and  $f$ , we prove, in particular, that there exists  $\lambda^* > 0$  such that, for each  $\lambda > \lambda^*$  and each Carathéodory function  $g$  with a sub-critical growth, the above problem has at least three weak solutions for every  $\mu \geq 0$  small enough.

**Keywords** Kirchhoff-type problem · Variational methods · Three critical points theorem

Let  $\Omega \subset \mathbf{R}^n$  be a bounded domain with smooth boundary, and let  $K : [0, +\infty[ \rightarrow \mathbf{R}$  be a given continuous function.

If  $n \geq 2$ , we denote by  $\mathcal{A}$  the class of all Carathéodory functions  $\varphi : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$  such that

$$\sup_{(x, t) \in \Omega \times \mathbf{R}} \frac{|\varphi(x, t)|}{1 + |t|^q} < +\infty,$$

where  $0 < q < \frac{n+2}{n-2}$  if  $n > 2$  and  $0 < q < +\infty$  if  $n = 2$ . While, when  $n = 1$ , we denote by  $\mathcal{A}$  the class of all Carathéodory functions  $\varphi : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$  such that, for each  $r > 0$ , the function  $x \rightarrow \sup_{|t| \leq r} |\varphi(x, t)|$  belongs to  $L^1(\Omega)$ .

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To Professor Franco Giannessi, with esteem, on his 75th birthday.

B. Ricceri (✉)

Department of Mathematics, University of Catania, Viale A. Doria 6, 95125, Catania, Italy  
e-mail: ricceri@dmf.unict.it

Given  $\varphi \in \mathcal{A}$ , consider the following Kirchhoff-type problem

$$\begin{cases} -K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \varphi(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A weak solution of this problem is any  $u \in H_0^1(\Omega)$  such that

$$K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \int_{\Omega} \nabla u(x) \nabla v(x) dx = \int_{\Omega} \varphi(x, u(x)) v(x) dx$$

for all  $v \in H_0^1(\Omega)$ .

We refer to [1–6, 8] for previous papers on this subject. There, in particular, the reader can find informations on its historical development, as well as the description of situations that can be realistically modeled by the previous problem with a non-constant  $K$ .

The aim of this paper is to establish the following result:

**Theorem 1** *Let  $f \in \mathcal{A}$ . Put*

$$\begin{aligned} \tilde{K}(t) &= \int_0^t K(s) ds \quad (t \geq 0), \\ F(x, t) &= \int_0^t f(x, s) ds \quad ((x, t) \in \Omega \times \mathbf{R}) \end{aligned}$$

and assume that the following conditions be satisfied:

- (a<sub>1</sub>)  $\sup_{u \in H_0^1(\Omega)} \int_{\Omega} F(x, u(x)) dx > 0$ ;
- (a<sub>2</sub>)  $\inf_{t \geq 0} \tilde{K}(t) > 0$ ;
- (a<sub>3</sub>) for some  $\alpha > 0$  one has

$$\liminf_{t \rightarrow +\infty} \frac{\tilde{K}(t)}{t^\alpha} > 0;$$

- (a<sub>4</sub>) there exists a continuous function  $h : [0, +\infty[ \rightarrow \mathbf{R}$  such that

$$h(tK(t^2)) = t$$

for all  $t \geq 0$

- (a<sub>5</sub>)  $\limsup_{t \rightarrow 0} \frac{\sup_{x \in \Omega} F(x, t)}{t^2} \leq 0$ ;
- (a<sub>6</sub>)  $\limsup_{|t| \rightarrow +\infty} \frac{\sup_{x \in \Omega} F(x, t)}{|t|^{2\alpha}} \leq 0$ .

Under such hypotheses, if we set

$$\theta^* = \inf \left\{ \frac{\tilde{K} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)}{2 \int_{\Omega} F(x, u(x)) dx} : u \in H_0^1(\Omega), \int_{\Omega} F(x, u(x)) dx > 0 \right\},$$

for each compact interval  $[a, b] \subset ]\theta^*, +\infty[$ , there exists a number  $r > 0$  with the following property: for every  $\lambda \in [a, b]$  and every  $g \in \mathcal{A}$  there exists  $\delta > 0$  such that, for each  $\mu \in [0, \delta]$ , the problem

$$\begin{cases} -K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in  $H_0^1(\Omega)$  are less than  $r$ .

To prove Theorem 1, we will use a corollary of a very recent result established in [7]. If  $X$  is a real Banach space, we denote by  $\mathcal{W}_X$  the class of all functionals  $\Phi : X \rightarrow \mathbf{R}$  possessing the following property: if  $\{u_n\}$  is a sequence in  $X$  converging weakly to  $u \in X$  and  $\liminf_{n \rightarrow \infty} \Phi(u_n) \leq \Phi(u)$ , then  $\{u_n\}$  has a subsequence converging strongly to  $u$ .

**Theorem A** [7, Theorem 2] *Let  $X$  be a separable and reflexive real Banach space;  $\Phi : X \rightarrow \mathbf{R}$  a coercive, sequentially weakly lower semicontinuous  $C^1$  functional, belonging to  $\mathcal{W}_X$ , bounded on each bounded subset of  $X$  and whose derivative admits a continuous inverse on  $X^*$ ;  $J : X \rightarrow \mathbf{R}$  a  $C^1$  functional with compact derivative. Assume that  $\Phi$  has a strict local minimum  $x_0$  with  $\Phi(x_0) = J(x_0) = 0$ . Finally, assume that*

$$\max \left\{ \limsup_{\|x\| \rightarrow +\infty} \frac{J(x)}{\Phi(x)}, \limsup_{x \rightarrow x_0} \frac{J(x)}{\Phi(x)} \right\} \leq 0$$

and that

$$\sup_{x \in X} \min\{\Phi(x), J(x)\} > 0.$$

Set

$$\sigma = \inf \left\{ \frac{\Phi(x)}{J(x)} : x \in X, \min\{\Phi(x), J(x)\} > 0 \right\}.$$

Then, for each compact interval  $[a, b] \subset \sigma, +\infty[$  there exists a number  $r > 0$  with the following property: for every  $\lambda \in [a, b]$  and every  $C^1$  functional  $\Psi : X \rightarrow \mathbf{R}$  with compact derivative, there exists  $\delta > 0$  such that, for each  $\mu \in [0, \delta]$ , the equation

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x)$$

has at least three solutions whose norms are less than  $r$ .

When we say that the derivative of  $\Phi$  admits a continuous inverse on  $X^*$  we mean that there exists a continuous operator  $T : X^* \rightarrow X$  such that  $T(\Phi'(x)) = x$  for all  $x \in X$ .

*Proof of Theorem 1* When  $n > 2$ , since  $f \in \mathcal{A}$ , for some  $p > 2$ , with  $p < \frac{2n}{n-2}$  if  $n > 2$ , we have

$$\sup_{(x,t) \in \Omega \times \mathbf{R}} \frac{|F(x,t)|}{1 + |t|^p} < +\infty. \tag{1}$$

Set

$$\beta = \begin{cases} 2\alpha & \text{if } n \leq 2 \\ 2 \min \left\{ \alpha, \frac{n}{n-2} \right\} & \text{if } n \geq 3. \end{cases}$$

Note that  $H_0^1(\Omega)$  is continuously embedded in  $L^\beta(\Omega)$ . Now, let us apply Theorem A taking  $X = H_0^1(\Omega)$ , with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}},$$

and, for each  $u \in X$ ,

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \tilde{K}(\|u\|^2), \\ J(u) &= \int_{\Omega} F(x, u(x)) dx. \end{aligned}$$

Clearly,  $\Phi$  is a sequentially weakly lower semicontinuous  $C^1$  functional which is bounded on each bounded subset of  $X$ , and  $J$  is a  $C^1$  functional with compact derivative (since  $f \in \mathcal{A}$ ). Moreover, since  $X$  is a Hilbert space and  $\tilde{K}$  is continuous and strictly increasing,  $\Phi$  belongs to the class  $\mathcal{W}_X$ , by a classical result. Let us show that  $\Phi'$  has a continuous inverse on  $X$  (we identify  $X$  to  $X^*$ ). To this end, let  $T : X \rightarrow X$  be the operator defined by

$$T(v) = \begin{cases} \frac{h(\|v\|)}{\|v\|} v & \text{if } v \neq 0 \\ 0 & \text{if } v = 0, \end{cases}$$

where  $h$  is the function appearing in  $(a_4)$ . Since  $h$  is continuous and  $h(0) = 0$ , the operator  $T$  is continuous in  $X$ . For each  $u \in X \setminus \{0\}$ , since  $K(\|u\|^2) > 0$  (by  $(a_2)$ ), we have

$$T(\Phi'(u)) = T(K(\|u\|^2)u) = \frac{h(K(\|u\|^2)\|u\|)}{K(\|u\|^2)\|u\|} K(\|u\|^2)u = \frac{\|u\|}{K(\|u\|^2)\|u\|} K(\|u\|^2)u = u,$$

as desired. Now, put

$$\gamma = \inf_{t \geq 0} K(t).$$

So,  $\gamma > 0$  (by  $(a_2)$ ) and

$$\tilde{K}(t) \geq \gamma t$$

for all  $t \geq 0$ . In particular, this implies that  $\Phi$  is coercive and 0 is the only global minimum of  $\Phi$ . Next, fix  $\epsilon > 0$ . In the sequel,  $c_i$  will denote positive constants independent of  $\epsilon$  and  $u \in X$ . By  $(a_5)$ , there is  $\eta > 0$  such that

$$F(x, t) \leq \epsilon t^2$$

for all  $(x, t) \in \Omega \times [-\eta, \eta]$ . If  $n = 1$ , due to the compact embedding of  $X$  into  $C^0(\overline{\Omega})$ , there is  $\delta_1 > 0$  such that, for every  $u \in X$  satisfying  $\|u\| < \delta_1$ , one has  $\sup_{\Omega} |u| < \eta$ , and so

$$J(u) \leq \epsilon \int_{\Omega} |u(x)|^2 dx \leq c_1 \epsilon \|u\|^2 \leq \frac{c_1 \epsilon}{\gamma} \tilde{K}(\|u\|^2)$$

from which

$$\limsup_{u \rightarrow 0} \frac{J(u)}{\Phi(u)} \leq \frac{2c_1 \epsilon}{\gamma}. \tag{2}$$

Now, assume  $n > 1$ . From (1), it easily follows that, for a suitable constant  $c_2 > 0$  one has

$$F(x, t) \leq c_2 |t|^p$$

for all  $(x, t) \in \Omega \times (\mathbf{R} \setminus [-\eta, \eta])$ . Consequently, we have

$$F(x, t) \leq \epsilon t^2 + c_2 |t|^p$$

for all  $(x, t) \in \Omega \times \mathbf{R}$ . So, by continuous embeddings, for a constant  $c_3 > 0$ , one has

$$J(u) \leq c_3 (\epsilon \|u\|^2 + \|u\|^p) \leq c_3 \left( \frac{\epsilon}{\gamma} \tilde{K}(\|u\|^2) + \left( \frac{\tilde{K}(\|u\|^2)}{\gamma} \right)^{\frac{p}{2}} \right)$$

for all  $u \in X$ . Consequently, since  $p > 2$ , we get

$$\limsup_{u \rightarrow 0} \frac{J(u)}{\Phi(u)} \leq \frac{2c_3 \epsilon}{\gamma}. \tag{3}$$

Now, put

$$\gamma_1 = \liminf_{t \rightarrow +\infty} \frac{\tilde{K}(t)}{t^\alpha}.$$

Then,  $\gamma_1 > 0$  (by  $(a_3)$ ) and, for a constant  $c_4 > 0$ , we have

$$\tilde{K}(t) \geq \gamma_1 t^\alpha - c_4 \tag{4}$$

for all  $t \geq 0$ . Observe also that, for a suitable  $M \in L^1(\Omega)$  (which is constant if  $n > 1$ ), we have

$$F(x, t) \leq \epsilon |t|^\beta + M(x) \tag{5}$$

for all  $(x, t) \in \Omega \times \mathbf{R}$ . Precisely, (5) follows from  $(a_6)$  when either  $n \leq 2$  or  $n \geq 3$  and  $\alpha < \frac{n}{n-2}$ . In the other case, it follows from (1). From (5), for a constant  $c_5 > 0$ , we then get

$$J(u) \leq c_5(\epsilon \|u\|^\beta + 1)$$

for all  $u \in X$ , and hence, if  $\|u\|$  is large enough, taking (4) into account, we have

$$\frac{J(u)}{\Phi(u)} \leq \frac{2c_5(\epsilon \|u\|^\beta + 1)}{\tilde{K}(\|u\|^2)} \leq \frac{2c_5(\epsilon \|u\|^\beta + 1)}{\gamma_1 \|u\|^{2\alpha} - c_4}.$$

Therefore, since  $\beta \leq 2\alpha$ , we have

$$\limsup_{\|u\| \rightarrow +\infty} \frac{J(u)}{\Phi(u)} \leq \frac{2c_5\epsilon}{\gamma_1}. \tag{6}$$

Since  $\epsilon$  is arbitrary, (2), (3) and (6) tell us that

$$\max \left\{ \limsup_{u \rightarrow 0} \frac{J(u)}{\Phi(u)}, \limsup_{\|u\| \rightarrow +\infty} \frac{J(u)}{\Phi(u)} \right\} \leq 0.$$

In other words, all the assumptions of Theorem A are satisfied. So, for each compact interval  $[a, b] \subset ]\theta^*, +\infty[$  there exists a number  $r > 0$  with the property described in the conclusion of Theorem A. Fix  $\lambda \in [a, b]$  and  $g \in \mathcal{A}$ . Put

$$\Psi(u) = \int_{\Omega} \left( \int_0^{u(x)} g(x, t) dt \right) dx$$

for all  $u \in X$ . So,  $\Psi$  is a  $C^1$  functional on  $X$  with compact derivative. Then, there exists  $\delta > 0$  such that, for each  $\mu \in [0, \delta]$ , the equation

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions in  $X$  whose norms are less than  $r$ . But the solutions in  $X$  of the above equation are exactly the weak solutions of the problem

$$\begin{cases} -K \left( \int_{\Omega} |\nabla u(x)|^2 dx \right) \Delta u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and the proof is complete. □

Now, some remarks on Theorem 1 follow.

*Remark 1* Observe that when  $n \geq 3$  and  $\alpha \geq \frac{n}{n-2}$ , condition  $(a_6)$  is automatically satisfied as  $f \in \mathcal{A}$ .

*Remark 2* When  $f$  does not depend on  $x$  and  $0$  is a local maximum for  $F$ , condition  $(a_5)$  is satisfied.

*Remark 3* Clearly, if the function  $K$  is non-decreasing in  $[0, +\infty[$ , with  $K(0) > 0$ , then the function  $t \rightarrow tK(t^2)$  ( $t \geq 0$ ) is increasing and onto  $[0, +\infty[$ , and so condition  $(a_4)$  is satisfied.

Next, we wish to point out a remarkable particular case of Theorem 1.

**Theorem 2** Let  $n \geq 4$ , let  $q \in ]0, \frac{n+2}{n-2}[$  and let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be a continuous function such that

$$\begin{aligned} \limsup_{|t| \rightarrow +\infty} \frac{|f(t)|}{|t|^q} &< +\infty, \\ \limsup_{t \rightarrow 0} \frac{F(t)}{t^2} &\leq 0, \\ \sup_{t \in \mathbf{R}} F(t) &> 0, \end{aligned}$$

where

$$F(t) = \int_0^t f(s) ds.$$

Then, if we fix  $a, b > 0$  and set

$$\theta^* = \inf \left\{ \frac{a \int_{\Omega} |\nabla u(x)|^2 dx + \frac{b}{2} \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^2}{2 \int_{\Omega} F(u(x)) dx} : u \in H_0^1(\Omega), \int_{\Omega} F(u(x)) dx > 0 \right\},$$

for each compact interval  $A \subset ]\theta^*, +\infty[$  there exists a number  $r > 0$  with the following property: for every  $\lambda \in A$  and every  $g \in \mathcal{A}$  there exists  $\delta > 0$  such that, for each  $\mu \in [0, \delta]$ , the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = \lambda f(u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in  $H_0^1(\Omega)$  are less than  $r$ .

*Proof* Fix  $a, b > 0$  and apply Theorem 1 taking

$$K(t) = a + bt$$

for all  $t \geq 0$ . Clearly,  $f \in \mathcal{A}$ . Condition  $(a_1)$  follows at once as  $\sup_{\mathbf{R}} F > 0$ . The validity of  $(a_2)$  and  $(a_5)$  is clear. Condition  $(a_4)$  holds for the reason pointed out in Remark 3. Finally, condition  $(a_3)$  holds with  $\alpha = 2$  and so, since  $2 \geq \frac{n}{n-2}$ , condition  $(a_6)$  is also satisfied, as noticed in Remark 1. The conclusion then follows directly from that of Theorem 1.  $\square$

*Remark 4* The already quoted very recent papers [3,5,6,8] are devoted to the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u(x)|^2 dx) \Delta u = \varphi(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely, [5,6] and [8] deal with the existence of three solutions of which one is positive, another is negative and the third one is sign-changing. The paper [3] deals with the existence of a sequence of positive solutions tending strongly to zero. It is not possible to do

a proper comparison between Theorem 2 and the results just quoted since both assumptions and conclusions are very different. For instance, let  $\varphi : \mathbf{R} \rightarrow \mathbf{R}$  be a continuous, non-negative and non-zero function whose support is compact and contained in  $]0, +\infty[$ . It is easy to see that no result from those papers can be applied to  $\varphi$ . Such a function, on the contrary, satisfies the assumptions of Theorem 2.

We conclude proposing two open problems.

Problem 1 Does the conclusion of Theorem 1 hold for each interval of the type  $]\theta^*, b]$  ?

Problem 2 Does Theorem 2 hold for  $n = 3$  ?

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